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Threshold singularities in a Fermi gas with attractive potential in one dimension

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Abstract

We consider the one-dimensional gas of fermions with spin S interacting via an attractive δ-function potential using the Bethe *Ansatz* solution. In zero magnetic field the atoms form bound states of $N = 2S + 1$ fermions, i.e. generalized Cooper states with each atom having a different spin component. For low energy excitations the system is a Luttinger liquid and is properly described by a conformal field theory with conformal charge $c = 1$. The linear dispersion of a Luttinger liquid is asymptotically exact in the low-energy limit where the band curvature terms in the dispersion are irrelevant. For higher energy excitations, however, the spectral function displays deviations in the neighborhood of the single-particle (hole) energy, which can be described by an effective X-ray edge type model. Using the Bethe *Ansatz* solution we obtain expressions for the critical exponents for the single-particle (hole) Green's function. This model can be relevant in the context of ultracold atoms with effective total spin S confined to an elongated optical trap.

1. Introduction

In a two- or higher-dimensional Fermi liquid the quasi-particle excitations are determined by the poles in the one-particle Green's function and their spectral weight. A Luttinger liquid

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distinguishes itself from a Fermi liquid by the absence of such quasi-particle poles in the low-energy regime. The pole is replaced in the single-particle spectral function by an asymmetric power-law singularity at the excitation energy with an exponent that depends on the interaction strength [1]. This is primarily due to the reduced phase space in 1D.

A Luttinger liquid is an interacting 1D Fermi gas with a dispersion that is linearized in the momentum about the Fermi points. Due to the conformal space–time invariance the power-law divergences of correlation functions are determined by the low-energy excitations of the system [2,3]. These excitations constitute the conformal towers in terms of four quantum numbers for each branch of excitations. The correlation functions calculated within conformal field theory are only asymptotically exact for long times and long distances because the curvature of the band has been neglected. The curvature terms in the Hamiltonian are formally irrelevant [4].

In a series of recent papers [5–13] it has been shown that neglecting the irrelevant operators corresponding to the curvature terms of the dispersion can in general lead to incorrect results for the threshold singularities in response functions. The corrective terms to be added to the Hamiltonian are of the form of a mobile impurity that is coupled to the Luttinger liquid modes. A nonperturbative treatment of the irrelevant operators then yields the threshold singularities in the one-particle (hole) Green's function correctly. The problem resembles the X-ray edge divergence in metals [14,15] that arises from the perturbation of the Fermi surface when a core electron is promoted (the impurity). The scattering phase shifts of the electrons off the impurity determine the exact critical exponent and this method is not limited to weak interactions. For integrable systems the phase shifts can be obtained from the Bethe *Ansatz* solution. This has been demonstrated for spinless fermions (anisotropic Heisenberg chain) [9], as well as for the Hubbard model (spinfull fermions) [13]. In this paper we address this issue for the case of a model for an ultracold gas of fermionic atoms of arbitrary spin S and attractive interaction potential in a 1D optical trap [16–20].

Previous work of a mobile impurity embedded into a Fermi gas should be pointed out [21–30]. Although these papers are closely related to the present work and Refs. [5–13], they refer to a foreign particle dragging through the Luttinger liquid. In the present case the "impurity" is just an excitation of the interacting 1D gas.

Spin-imbalanced ultracold gases of fermionic atoms confined to 1D traps have been the subject of several recent studies [16,31,32]. Confinement to nearly 1D tubes can be achieved if the ultracold cloud of atoms is subjected to a 2D optical lattice, which defines a 2D array of tubes [31]. The tubes can be regarded as isolated if the confinement by the laser beams is strong enough to suppress tunneling between tubes. The scattering between atoms under transverse harmonic confinement is subject to a confinement-induced resonance [33]. Fine-tuning this Feshbach-type resonance, the interaction between particles can be made attractive and its strength can be varied [34]. The interaction is local and can be approximated by a δ-function potential in space. The model is integrable and has been solved via Bethe's *Ansatz* [35–40].

For fermions with a spin S an attractive potential may lead to bound states of up to $N = 2S + 1$ atoms, extending the concept of preformed Cooper pairs to larger clusters. The ground state phase diagram will have large number of possible pure (consisting of only one kind of clusters) and mixed phases (coexistence of two or more basic states) [18,40]. In zero magnetic field only bound states of N atoms (one atom with each spin component) can exist. All other bound states are gapped. In the Luttinger liquid limit the model then corresponds to a conformal field theory with central charge $c = 1$. The response functions determining the superfluid and density wave order have been calculated using conformal field theory and the exact Bethe *Ansatz* solution [50]. In the absence of a magnetic field, for $S = 1/2$ superfluidity is a possibility for

all densities and density waves can be excluded. For $S \geq 3/2$ superfluidity may occur at low densities but at high densities it gives way to density waves [55]. A possible application for $S = 3/2$ is the alpha-particle condensation in nuclear matter. The four degrees of freedom are made up of the spin and pseudospin degrees of freedom. The alpha particles are strongly bound and exist in a Bose–Einstein condensation phase [56]. Here we address the threshold singularity of the one-particle and one-hole (particles and holes refer to bound states) Green's function for high energy excitations where the dispersion is no longer linear in the momentum.

The remainder of the paper is organized as follows. In Section 2 we restate the model, the classification of states of the Bethe *Ansatz* equations, the integral equations satisfied by the dressed energy and density of states for bound states of N atoms in zero magnetic field, the elementary excitations and the conformal towers of the finite size excitations. We also briefly summarize the thermodynamic properties of the system. In Section 3 we address the effective field-theoretical model with the high-energy mode (mobile impurity term), the one-particle (hole) spectral function and the anomalous exponents from the phase shifts. In Section 4 we obtain the finite size corrections to the ground state energy including the high-energy mode through the Bethe *Ansatz*. The finite size terms are then related to the scattering phase shifts. Conclusions follow as Section 5.

2. Model and Bethe *Ansatz* equations

2.1. The model

The Hamiltonian for a gas of nonrelativistic particles with $(2S+1)$ colors (spin S) interacting via an attractive δ-function potential is

$$\mathcal{H} = -\sum_{i=1}^{N_t} \frac{\partial^2}{\partial x_i^2} - 2|c| \sum_{i<j} \delta(x_i - x_j), \tag{1}$$

where x_i are the coordinates, N_t is the total number of particles and c is the interaction strength. By fine-tuning the confinement-induced resonance in a gas of fermionic cold atoms [33] the interaction can become attractive and its strength can be varied. Here $\hbar^2/2m$, where m is the mass of the particles, has been equated to 1, or alternatively it has been scaled into $\mathcal{H}$ and c. The length of the trap is L.

2.2. Classification of Bethe states

The states of the coordinate Bethe *Ansatz* are plane waves constructed from the two-particle scattering matrix. This scattering matrix satisfies the so-called Yang–Baxter triangular relation, which is a necessary condition for integrability [35,36]. As a consequence of the triangular relation many-particle scattering processes can be factorized into two-particle processes and the order in which the individual scattering processes take place can be interchanged (the order becomes arbitrary). A complete solution can be constructed by iteratively applying (nesting) $N = 2S + 1$ Bethe *Ansätze*, such that one color is eliminated at each step [37]. Each Bethe *Ansatz* gives rise to a new set of rapidities, so that the solution consists of N sets of rapidities. All rapidities within a given set have to be different to ensure linearly independent solutions giving rise to Fermi statistics of the states.

The classification of states is completely determined by the SU(N)-invariance of the model and the attractive effective potential. The classification of states is therefore completely analogous to that of the Anderson impurity of arbitrary spin in the $U \to \infty$ limit [41–45] and the one-dimensional degenerate supersymmetric t–J model [46,47]. The interaction strength in the case of the Anderson impurity model is determined by the hybridization, while the supersymmetric t–J model has no energy scale and the interaction strength is equal to one. Since only particles with different spin components experience an effective attractive interaction, we may build bound states of up to $(2S+1)$ particles. In zero magnetic field the number of particles of each spin-component (color) is the same, i.e. there is no spin imbalance and it is then energetically favorable to bind all the particles into clusters of N atoms. Each cluster has one atom of each color. These clusters are the generalization of preformed Cooper pairs to N colors. Preformed Cooper pairs are Cooper pairs that exist above the transition temperature, which in 1D is $T_c = 0$. Hence, these clusters of N colors exist even at finite T (see Refs. [18,39,49,38,48]).

The string solutions of the Bethe *Ansatz* equations in a system with finite length have finite size corrections of the order $\mathcal{O}(e^{-\kappa L})$, where κ is a positive constant [48]. These exponentially small corrections can be neglected in comparison to the mesoscopic terms that are of order $1/L$.

2.3. Dressed energy potential in zero magnetic field

In zero magnetic field and for the ground state the rapidities for the N-particle clusters, ξ, are densely distributed in the interval $[-Q, Q]$. Only one dressed energy potential $\varepsilon(\xi)$ (entering the Fermi–Dirac distribution) needs to be considered, which satisfies the integral equation [40,49,50]

$$\varepsilon(\xi) = D_S(\xi) - \int_{-Q}^{Q} d\xi' K_S\big(\xi - \xi'\big)\varepsilon\big(\xi'\big), \tag{2}$$

where the driving term $D_S(\xi)$ is given by

$$D_S(\xi) = (2S+1)\left[\xi^2 - \frac{S(S+1)}{3}c^2 - \mu\right], \tag{3}$$

and the integration kernel $K_S(\xi)$ is

$$K_S(\xi) = \sum_{l=1}^{2S} \frac{1}{\pi}\frac{l|c|}{\xi^2 + (lc)^2}. \tag{4}$$

Here μ is the chemical potential, playing the role of the Lagrange parameter for the conservation of the total number of bound states. μ determines the integration limit Q through the condition that $\varepsilon(\pm Q) = 0$. Here Q determines the two Fermi points of the system, since occupied states correspond to $\varepsilon(\xi) < 0$ and for empty states $\varepsilon(\xi)$ is positive. Since bound states of N particles have one particle with each spin component, these clusters do not have magnetic response and Eq. (2) does not depend on the magnetic field (assumed to be zero or less than the binding energy of the bound states). If ε is rescaled to ε/c^2, μ to μ/c^2, Q to $Q/|c|$ and ξ to $\xi/|c|$, Eq. (2) is universal, i.e., independent of the magnitude of $|c|$. Hence, within the framework of the grand canonical ensemble, without loss of generality, it is sufficient to present the results for $|c| = 1$.

2.4. Distribution density of rapidities

The density function of the rapidities is obtained from the dressed energy $\varepsilon(\xi)$ by differentiation with respect to μ, i.e. [40,49,50],

$$\rho_h(\xi) + \rho(\xi) = -\frac{1}{2\pi}\frac{\partial \varepsilon(\xi)}{\partial \mu}, \tag{5}$$

where $\rho(\xi)$ is the particle density and $\rho_h(\xi)$ the corresponding hole density for bound states involving $2S+1$ particles. The integral equation satisfied by the density function is

$$\rho_h(\xi) + \rho(\xi) = \frac{2S+1}{2\pi} - \int_{-Q}^{Q} d\xi' K_S\big(\xi - \xi'\big)\rho\big(\xi'\big), \tag{6}$$

where the integration kernel and the integration limits are the same ones as for the dressed energy potential, but the driving term is different [48].

The total number of atoms per unit length (i.e. $(2S+1)$ times the number of bound states) is given by

$$N_t/L = n = \int_{-Q}^{Q} d\xi \rho(\xi). \tag{7}$$

If ξ and Q are rescaled to $\xi/|c|$ and $Q/|c|$, respectively, as discussed above, then ρ remains invariant and n rescales to $|c|n$.

The ground state energy of the system is given by

$$E_{GS} = L\epsilon_\infty = L\int_{-Q}^{Q} d\xi \left[\xi^2 - \frac{S(S+1)}{3}c^2\right]\rho(\xi), \tag{8}$$

where ϵ_∞ is the ground state energy density.

2.5. Particle and hole excitations

The gas of fermions with attractive δ-function interaction has several classes of excitations. Only one class of excitations is gapless, namely, excitations within the band of bound states of $(2S+1)$ particles. The bands of bound states of less than $(2S+1)$ particles are all gapped in zero magnetic field. String excitations correspond to bound states of clusters and are also gapped. Hence, for the present purpose we need to consider only the band of bound states of $(2S+1)$ particles.

Particle (hole) excitations are obtained by adding (removing) a bound state of N atoms to (from) the system. Let us denote the rapidity of added or removed bound state with ξ_0. The energy of the elementary excitation is given by the dressed energy potential $\varepsilon(\xi_0)$. Particle excitations correspond to $|\xi_0| > Q$ and the energy is $\varepsilon_p(\xi_0) = \varepsilon(\xi_0)$, while hole excitations to $|\xi_0| < Q$ and the energy is $\varepsilon_h(\xi_0) = -\varepsilon(\xi_0)$. The Fermi points are given by $\xi_0 = \pm Q$ and the energy vanishes at the Fermi points because $\varepsilon(\pm Q) = 0$. The momentum of the excitation is given by

$$p_{p,h}(\xi_0) = 2\pi \int_{0}^{\xi_0} d\xi \left[\rho(\xi) + \rho_h(\xi)\right]. \tag{9}$$

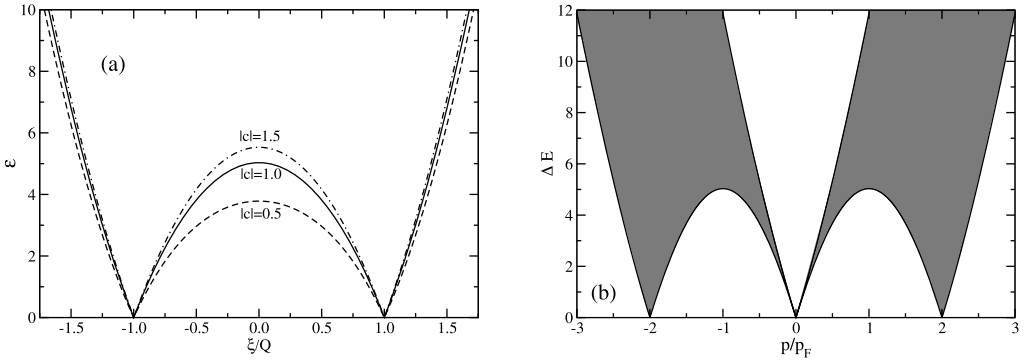

Fig. 1. (a) Elementary particle and hole excitations as a function of the rapidity for $Q = 1$, $S = 5/2$ and three values of the interaction strength, $|c| = 1.0$ (solid curve), $|c| = 0.5$ (dashed curve) and $|c| = 1.5$ (dash-dotted). The particle excitations correspond to $|\xi| > Q$ and the hole excitations to $|\xi| < Q$. $\xi = \pm Q$ corresponds to the Fermi points. (b) Particle–hole continuum (shaded area) for $Q = 1$, $S = 5/2$ and $|c| = 1.0$ as a function of the momentum.

This leads to the Fermi momentum $p_F = \pi n$, which is completely determined by the occupation of the band. The low-energy excitation states form a simple Dirac sea with two Fermi points at $\xi = \pm Q$ or $p = \pm p_F$. In Fig. 1(a) we show the dispersion of the excitations and the shaded area in Fig. 1(b) displays the continuum of particle–hole excitations, which is generated as the linear superposition of a particle and a hole excitation.

The group velocity for bound states of N particles is defined as [40,49]

$$v_F = \left(\frac{d\varepsilon(\xi)}{d\xi}\bigg|_{\xi=Q}\right)\Big/\big(2\pi\rho(Q)\big). \tag{10}$$

This group velocity is the slope in Fig. 1(a) of the low-energy states and corresponds to the Fermi velocity. The group velocity away from the linear dispersion regime is

$$u = \left(\frac{d\varepsilon(\xi_0)}{d\xi_0}\right)\Big/\big(2\pi\big(\rho(\xi_0) + \rho_h(\xi_0)\big)\big) \tag{11}$$

and corresponds to the slope of the dispersion in Fig. 1(a).

An alternative way to obtain the excitation energy (yielding the same result) is through Eq. (6) [49]. We first consider the case of holes. Removing a rapidity $|\xi_0| < Q$ introduces an additional driving term in Eq. (6) which is of the order of $1/L$ smaller than the main driving term. The solution is now of the form $\rho(\xi) + \Delta\rho(\xi, \xi_0)/L$, where $\rho(\xi)$ is the ground state distribution and $\Delta\rho(\xi, \xi_0)$ accounts for the rearrangement of the momenta due to the missing one. $\Delta\rho(\xi, \xi_0)$ satisfies the following integral equation

$$\Delta\rho(\xi, \xi_0) = -\delta(\xi - \xi_0) - \int_{-Q}^{Q} d\xi' K_S\big(\xi - \xi'\big)\Delta\rho\big(\xi', \xi_0\big), \tag{12}$$

where the first term on the right-hand side is the missing rapidity. We write $\Delta\rho(\xi, \xi_0) = -\delta(\xi - \xi_0) + \Delta\rho_1(\xi, \xi_0)$, so that $\Delta\rho_1(\xi, \xi_0)$ satisfies

$$\Delta\rho_1(\xi, \xi_0) = K_S(\xi - \xi_0) - \int_{-Q}^{Q} d\xi' K_S\big(\xi - \xi'\big)\Delta\rho_1\big(\xi', \xi_0\big). \tag{13}$$

Here the integration limit Q has to be readjusted, since the integral over $\Delta\rho(\xi,\xi_0)$ is not equal -1. The energy of the excitation is

$$\Delta E_h(\xi_0) = -(2S+1)\left[\xi_0^2 - \frac{S(S+1)}{3}c^2\right] + (2S+1)\int_{-Q}^{Q} d\xi'\left[\xi'^2 - \frac{S(S+1)}{3}c^2\right]\Delta\rho_1\big(\xi',\xi_0\big) + \big(E(Q) - E(Q_0)\big), \tag{14}$$

where the last term accounts for the change in the integration limit

$$E(Q) - E(Q_0) = (2S+1)\left(1 - \int_{-Q}^{Q} d\xi\,\Delta\rho_1(\xi)\right)\mu. \tag{15}$$

For particle excitations we add a rapidity $|\xi_0| > Q$, which changes the distribution of rapidities by the amount of $\Delta\rho_2(\xi,\xi_0)/L$,

$$\Delta\rho_2(\xi,\xi_0) = K_S(\xi-\xi_0) - \int_{-Q}^{Q} d\xi' K_S\big(\xi-\xi'\big)\Delta\rho_2\big(\xi',\xi_0\big). \tag{16}$$

Comparing Eqs. (13) and (16) we have that $\Delta\rho_2(\xi,\xi_0) = -\Delta\rho_1(\xi,\xi_0)$, except that in one case $|\xi_0| > Q$ and in the other $|\xi_0| < Q$. The energy of the particle excitation is

$$\Delta E_p(\xi_0) = (2S+1)\left[\xi_0^2 - \frac{S(S+1)}{3}c^2 - \mu\right] + (2S+1)\int_{-Q}^{Q} d\xi'\left[\xi'^2 - \frac{S(S+1)}{3}c^2 - \mu\right]\Delta\rho_2\big(\xi',\xi_0\big), \tag{17}$$

where the terms involving μ correspond to the readjustment of Q to comply with the conservation of states. Except for the overall sign, expressions (14) and (17) are the same, although one is valid for $|\xi_0|$ smaller than Q and the other for $|\xi_0| > Q$. Hence, for $\xi_0 = \pm Q$ the excitation energy vanishes, corresponding to the "Fermi surface". $\Delta E_h(\xi_0)$ and $\Delta E_p(\xi_0)$ are identical to $|\varepsilon(\xi_0)|$.

2.6. Conformal towers

The Luttinger liquid limit of the model corresponds to the excitation spectrum linearized in the momentum about the Fermi points with Fermi velocity v_F. The low-lying excitations are described by the finite size corrections to the ground state energy in the Luttinger liquid [2,3,57]. While the ground state energy for the bulk, E_{GS} given by Eq. (8), is an extensive quantity, the mesoscopic corrections are of order $1/L$, where L is the length of the system. Hence, the finite size corrections arising from the string solutions for the bound states, which are of the order of $\mathcal{O}(e^{-\kappa L})$ with $\kappa > 0$, can be neglected [48]. For periodic boundary conditions, the finite size corrections of the populated Dirac sea of bound states of N atoms are determined by four quantum numbers, namely ΔM corresponds to the added or removed number of bound states

(rapidities), D is the parity variable, i.e. $2D$ is the difference between forward and backward movers, and the $n^{\pm}$ count the number of particle and hole excitations about each Fermi point ("+" for forward movers and "−" for backward movers). The finite size correction for the present model in the absence of a magnetic field have been discussed in Ref. [50].

The finite size corrections to the ground state energy are given by [2,3,13,18,57,58]

$$E = E_{GS} + \frac{2\pi v_F}{L}\left\{\frac{1}{4}[\Delta M/z]^2 + [zD]^2 + n^+ + n^- - \frac{1}{12}\right\} \tag{18}$$

and the corresponding change in momentum for the excitations is

$$\Delta P = \frac{2\pi}{L}\left[D\Delta M + n^+ - n^-\right] + p_F D. \tag{19}$$

In Eq. (18) z is the generalized dressed charge for the band. The generalized dressed charge determines how the two Fermi points interact with each other, i.e. the way a change of a quantum number ΔM or D affects the contribution to the energy. For elementary excitations from the ground state, the values of the D quantum number are constrained by the discrete Bethe *Ansatz* equations for periodic boundary conditions. A change in the population, ΔM, changes the backscattering quantum number by

$$D = \tfrac{1}{2}\Delta M \ (\text{mod } 1). \tag{20}$$

Note that D is only determined modulo 1.

The dressed generalized charge is obtained as $z = \Xi(Q)$, where the Ξ is the solution of the integral equation

$$\Xi(\xi) = 1 - \int_{-Q}^{Q} d\xi' K_S(\xi - \xi')\Xi(\xi'). \tag{21}$$

Here the integration kernel is the same as for the integral equation for $\varepsilon(\xi)$, i.e. Eq. (4). Comparing Eqs. (6) and (21) it follows that $\Xi(\xi) = [2\pi/(2S+1)][\rho_h(\xi) + \rho(\xi)]$, so that $z = [2\pi/(2S+1)]\rho(Q)$. Note that z is always nonnegative. z is a monotonically decreasing function of $Q/|c|$ and is shown in Fig. 2(a) for various spin values. Note that the dependence of z with the integration limit increases dramatically with increasing S.

For very large Q the Fredholm integral equation (21) can be reduced to a Wiener–Hopf integral equation, by shifting the variable ξ by the amount Q. The Wiener–Hopf equation can be solved analytically with standard methods (see, e.g., Appendix B of [45]). This way we obtain asymptotically $z(Q \to \infty) = 1/\sqrt{2S+1}$.

In a magnetic field up to $N = 2S+1$ bands can be partially filled and contribute to the conformal towers. The dressed generalized charges now form in general a $N \times N$ matrix and there are four quantum numbers for each band. The cases $S = 1/2$ [13,19,58], $S = 3/2$ [19] and $S = 5/2$ [20] have been extensively investigated previously.

2.7. *Physical properties for ultracold atoms: Previous results*

So far experimental efforts were mostly focused on ^{6}Li, i.e. a $S = 1/2$ system [31,51–53]. Tubes with ultracold gases of fermionic atoms provide the unique possibility to study systems with a spin larger than 1/2, e.g., ^{40}K (spin 9/2), ^{43}Ca (spin 7/2), ^{87}Sr (spin 9/2), ^{173}Yb (spin 5/2), ^{9}Be (spin 3/2), ^{135}Ba (spin 3/2), ^{137}Ba (spin 3/2) and ^{53}Cr (spin 3/2) atoms [54].

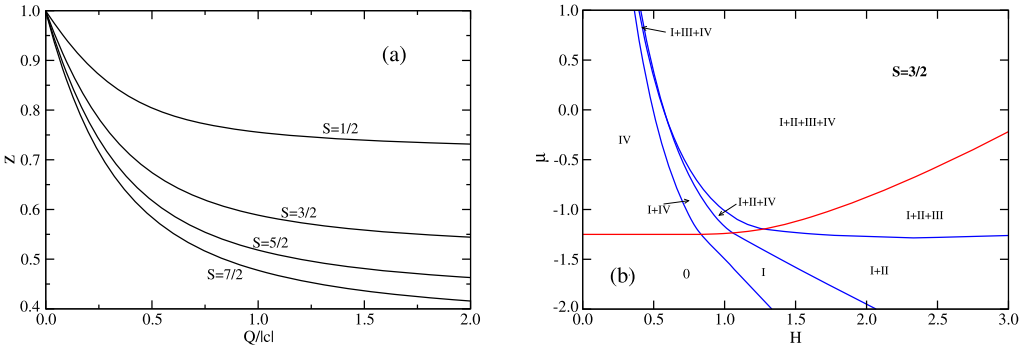

Fig. 2. (a) Generalized dressed charge z as a function of $Q/|c|$ for several spin values. (b) Ground state phase diagram of the chemical potential μ vs. the magnetic field H for a homogeneous fermion gas of spin $S = 3/2$ with $|c| = 1$. The Roman numbers denote the number of particles involved in a bound state. Regions with more than one Roman number are mixed phases with coexisting bound states. The lower left corner (0) corresponds to the empty system (no particles).

In zero magnetic field the atoms form bound states of $(2S + 1)$ particles, one of each spin component, which are the generalization of preformed Cooper pairs to larger spins. For an isolated tube there is no long-range order of the bound states. A weakly interacting array of tubes, e.g., via tunneling of bound states between tubes, however, increases the effective dimension of the system so that long-range order can arise [60,61]. Two types of order has been considered: superfluidity and density waves [50]. The superfluid order parameter will be homogeneous (not modulated) and the critical exponent of the superfluid response function increases with the particle density. On the other hand, the exponent of the particle density wave response function decreases with the particle density and for $S \geq 3/2$ density waves are the dominating form of order at high densities [50,55,56].

In optical one-dimensional traps the weak confinement of the atoms along the tube caused by the laser beams is roughly harmonic and can be locally incorporated into the chemical potential [16]. Consequently, the system is only locally homogeneous and displays phase separation with the variation of the chemical potential along the tube [16,32]. At the trap boundaries there is a higher chance of superfluidity for $S \geq 3/2$ than the center of the trap, which is more prone to density waves [50].

In a finite magnetic field bound states of less than N atoms become possible, giving rise to a complicated phase diagram of phases with coexisting clusters. Bound states with less than N atoms carry a magnetic moment and their population can be controlled by the chemical potential μ and the magnetic field H. Hence, a rich μ versus H ground state phase diagram arises for spins $S = 3/2,\ 5/2,\ 7/2$ and $9/2$ [17,18], extending Orso's [16] work for $S = 1/2$ (see Fig. 2(b) for $S = 3/2$). Two types of inhomogeneous phases of may arise [18–20]: (a) phase separation along the tube, and (b) modulations of the superfluid order parameters of the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) type [62].

In case (a), the confining harmonic potential varies with the position along the tube, so that the chemical potential is a function of position and hence different phases for a given magnetic field appear along the trap giving rise to phase separation [16,18,31,32,20]. Bound states of $(2S + 1)$ particles predominate at the center of the trap, while spin-paired and spin-polarized atoms dominate at the fringes of the trap.

In case (b), there is a very large number of possible order parameters for superfluidity for the spin-imbalanced gas [19,20]. As for unconventional superconductors, the instability to

superfluidity from the normal phase is determined by the properties of the Luttinger liquid. The corresponding correlation functions have been studied in detail for $S = 3/2$ and $S = 5/2$ using conformal field theory and the Bethe *Ansatz* solution [19,20]. A combination of three criteria determine the dominating order in the case of a spin imbalanced gas: (i) the smallest critical exponent corresponds to the longest range and is hence favorable to order if it is less than 2, the critical scaling dimension in 1D, (ii) a large distance between nodes favors order because strong variations of the order parameter are energetically less unfavorable, and (iii) the preformed bound states should carry a small momentum, since a large momentum of the bound states is energetically unfavorable to a condensate. Cooper-*pairing* of atoms with spin-component S and $S-1$ (one particle being a forward mover and the other a backward mover) is found to be the first superfluid phase accessed from the normal state as the temperature is lowered in a sufficiently large field. Due to the spin-imbalance of the atoms in a magnetic field inhomogeneities like modulations of the order parameter of the FFLO type will arise. There is no long-range order in a strictly 1D system. Long-range order in general requires an array of tubes with interactions between particles in different tubes or Josephson tunneling between tubes [19,59,60] giving rise to a dimensional crossover from 1D to 3D, opening the door to the possibility of superfluid long-range order at finite temperature. Note that the two main conditions for realization of the FFLO phase are satisfied in cold atom tubes: (1) the system is very pure (no impurities) and (2) it has a low effective dimension (extreme anisotropy).

3. Field theory model for the Luttinger liquid with mobile impurity

The model with a dispersion linearized in the momentum is a spinless Luttinger liquid [4] which can be parametrized by a Bose field $\Phi(x)$ and its dual field $\Theta(x)$ that satisfy the commutation relations

$$\left[\Phi(x), \frac{\partial\Theta(y)}{\partial y}\right] = i\pi\delta(x-y). \tag{22}$$

The Hamiltonian of the Luttinger liquid is

$$\mathcal{H}_{LL} = \frac{v_F}{2\pi}\int dx\left[\frac{1}{K}\left(\frac{\partial\Phi(x)}{\partial x}\right)^2 + K\left(\frac{\partial\Theta(x)}{\partial x}\right)^2\right], \tag{23}$$

where we have neglected irrelevant operators. K is known as the Luttinger parameter parametrizing the interaction strength. For the noninteracting system $K = 1$. For a system confined to a box of length L, the excitation spectrum depends on the boundary conditions of the fields $\Phi(x)$ and $\Theta(x)$ (see Eq. (18)). Within the framework of the Luttinger model, i.e. neglecting irrelevant terms in Eq. (23), the spectral function for the retarded one-particle Green's function exhibits singularities at $\omega = v_F p$ [11]

$$A(p,\omega) \propto (\omega - v_F p)^{\mu_{LL}}\theta(\omega - v_F p), \quad \mu_{LL} = \frac{1}{4}\left(K + \frac{1}{K} - 2\right) - 1, \tag{24}$$

where the Fermi point $+p_F$ has been considered. Similar expressions hold for the one-hole Green's function and for the Fermi point for backward movers. The physical reason for this singularity is that adding (or removing) a fermion in the Luttinger liquid creates multiple low-energy excitations in the system [1].

It has been demonstrated for spinless Luttinger liquids [5–11], as well as for spinfull 1D models [12,13], that neglecting the irrelevant operators arising from the nonlinear dispersion leads in general to incorrect results in the threshold position and the exponents in response functions. A high energy excitation perturbing the Luttinger Hamiltonian can be included by coupling the Luttinger liquid to a mobile impurity. Treating this impurity nonperturbatively leads to the exact singularities in the response function. The exponents are in general momentum dependent.

Adding a particle with energy $\varepsilon_p(p)$ is mapped onto the following mobile impurity Hamiltonian (see, e.g., Refs. [8,9,11–13])

$$\mathcal{H}_d = \int dx d^\dagger(x)\left[\varepsilon_p(p) - iu\frac{\partial}{\partial x}\right]d(x), \tag{25}$$

where $d(x)$ and $d^\dagger(x)$ are the annihilation and creation operators of the mobile impurity, u is the group velocity of the excitation and p the momentum. The interaction of the mobile impurity with the Luttinger liquid is given by

$$\mathcal{H}_{int} = \int dx\left[\frac{V_L - V_R}{2\pi}\frac{\partial\Theta(x)}{\partial x} + \frac{V_L + V_R}{2\pi}\frac{\partial\Phi(x)}{\partial x}\right]d^\dagger(x)d(x), \tag{26}$$

where V_R and V_L are coupling parameters. In Section 4 we relate the parameters in Eqs. (23), (25) and (26) with quantities obtained from the Bethe *Ansatz*.

Considering $\mathcal{H} = \mathcal{H}_{LL} + \mathcal{H}_{imp} + \mathcal{H}_{int}$ we now apply a canonical transformation U to all operators to eliminate the terms linear in the fields $\partial_x\Theta$ and $\partial_x\Phi$ [9,13],

$$U = \exp\left\{-\frac{i}{2\pi}\int dx\left[-(\varphi_+ - \varphi_-)\Theta(x) + \frac{\varphi_+ + \varphi_-}{K}\Phi(x)\right]d^\dagger(x)d(x)\right\}, \tag{27}$$

where φ_+ and φ_- are parameters to be determined. Denoting the transformed quantities $\bar{d} = UdU^\dagger$, $\overline{\Phi} = U\Phi U^\dagger$ and $\overline{\Theta} = U\Theta U^\dagger$ we have

$$\begin{aligned}
\partial_x\Theta &= \partial_x\overline{\Theta} - \frac{1}{2K}(\varphi_+ + \varphi_-)\bar{d}^\dagger\bar{d},\\
\partial_x\Phi &= \partial_x\overline{\Phi} + \frac{1}{2}(\varphi_+ - \varphi_-)\bar{d}^\dagger\bar{d},\\
d &= \bar{d}\exp\left\{-\frac{i}{2\pi}\left[-(\varphi_+ - \varphi_-)\overline{\Theta} + \frac{1}{K}(\varphi_+ + \varphi_-)\overline{\Phi}\right]\right\}.
\end{aligned} \tag{28}$$

The unwanted terms disappear if [9,13]

$$\begin{aligned}
(V_L - V_R) &= (v-u)\varphi_+ + (v+u)\varphi_-,\\
(V_L + V_R)K &= -(v-u)\varphi_+ + (v+u)\varphi_-
\end{aligned} \tag{29}$$

and the transformed Hamiltonian becomes noninteracting

$$\begin{aligned}
\mathcal{H}_{tran} = \frac{v_F}{2\pi}\int dx\left[\frac{1}{K}\left(\frac{\partial\overline{\Phi}(x)}{\partial x}\right)^2 + K\left(\frac{\partial\overline{\Theta}(x)}{\partial x}\right)^2\right]\\
+ \int dx\bar{d}^\dagger(x)\left[\varepsilon_p(p) - iu\frac{\partial}{\partial x}\right]\bar{d}(x),
\end{aligned} \tag{30}$$

but at the expense of boundary terms for the boson operators. Taking expectation values in Eq. (28) we have [9]

$$\Delta M = \frac{1}{\pi}\int_0^L dx\,\langle\partial_x \Phi\rangle = \frac{1}{\pi}\int_0^L dx\,\langle\partial_x \overline{\Phi}\rangle - \frac{1}{2\pi}(\varphi_+ - \varphi_-),$$
$$2D = -\frac{1}{\pi}\int_0^L dx\,\langle\partial_x \Theta\rangle = -\frac{1}{\pi}\int_0^L dx\,\langle\partial_x \overline{\Theta}\rangle + \frac{1}{2\pi K}(\varphi_+ + \varphi_-), \tag{31}$$

where $\Delta M = 1$ represents the change in the number of particles and D is the backscattering quantum number (charge and current quantum numbers of the excitation).

Using Eq. (28) the spectral function of the one-particle Green's function, $G_p(p,t) \propto \int dx\,\langle d(x,t)d^\dagger(0,0)\rangle$, is now given by [9,11]

$$A_p(p,\omega) \propto \begin{cases} \frac{\theta(\omega-\varepsilon_p)\sin(\pi\mu_L)+\theta(\varepsilon_p-\omega)\sin(\pi\mu_R)}{|\omega-\varepsilon_p|^{1-\mu}} & \text{if } u > v_F, \\ \frac{\theta(\omega-\varepsilon_p)\sin(\pi\mu)}{|\omega-\varepsilon_p|^{1-\mu}} & \text{if } u < v_F, \end{cases} \tag{32}$$

where $\mu = \mu_R + \mu_L$ and

$$\mu_R(p) = \frac{1}{K}\left(\frac{\varphi_+(p)}{2\pi}\right)^2, \qquad \mu_L(p) = \frac{1}{K}\left(\frac{\varphi_-(p)}{2\pi}\right)^2. \tag{33}$$

Hence, in contrast to the Luttinger liquid result, Eq. (24), the threshold of the singularity is now at the excitation energy $\varepsilon_p(p)$ and the exponent is momentum dependent. The spectral function is asymmetric about $\omega = \varepsilon_p$. The $\varphi_\pm$ are the scattering phase shifts for the forward and backward movers, respectively.

Similar results can be obtained for a hole excitation. For hole excitations one expects that $u < v_F$ and the spectral function in this case is [9,11]

$$A_h(p,-\omega) \propto \frac{\theta(\omega-\varepsilon_h)\sin(\pi\mu)}{|\omega-\varepsilon_h|^{1-\mu}}, \tag{34}$$

where $\varepsilon_h > 0$.

4. Relation to the Bethe *Ansatz* results

Using the Bethe *Ansatz* equations we now calculate the finite size corrections to the ground state energy of our system in the presence of a high energy excitation. We consider a hole excitation, but the results for a particle excitation are similar. An excitation introduces a small asymmetry in the integration limits, which we now denote by Q_+ and Q_-. Without the excitation we had $Q_+ = Q$ and $Q_- = -Q$.

Following Refs. [2] and [13] the starting point are the discrete Bethe *Ansatz* equations, $\mathcal{Z}(\xi_j) = -2\pi I_j/L$, with

$$\mathcal{Z}(\xi) = (2S+1)\xi - \sum_i \sum_{l=1}^{2S} \frac{2}{L}\arctan\left(\frac{\xi-\xi_i}{l|c|}\right), \tag{35}$$

which using the Euler–MacLaurin sum formula for large L is transformed into the integral form

$$\mathcal{Z}(\xi) = (2S+1)\xi - \int_{Q_-}^{Q_+} d\xi' \rho(\xi') \sum_{l=1}^{2S} 2\arctan\left(\frac{\xi-\xi'}{l|c|}\right)$$
$$+ \frac{2}{L}\sum_{l=1}^{2S}\arctan\left(\frac{\xi-\xi_L^h}{l|c|}\right) + \frac{1}{12L^2}\frac{1}{2\pi\rho(Q_+)}\sum_{l=1}^{2S}\frac{l|c|}{(\xi-Q_+)^2+(lc)^2}$$
$$- \frac{1}{12L^2}\frac{1}{2\pi\rho(Q_-)}\sum_{l=1}^{2S}\frac{l|c|}{(\xi-Q_-)^2+(lc)^2} + \mathcal{O}\left(L^{-3}\right). \tag{36}$$

Here the term with ξ_L^h is the term of the excitation with the missing quantum number I^h, $\mathcal{Z}(\xi_L^h) = 2\pi I^h/L$. Differentiating with respect to ξ and dividing by 2π we obtain the integral equation for ρ. Expanding ρ in powers of $1/L$, i.e. $\rho_0(\xi) + \rho_1(\xi)/L + \rho_2(\xi)/L^2 + \mathcal{O}(L^{-3})$, we have

$$\rho_0(\xi) = \frac{2S+1}{2\pi} - \int_{Q_-}^{Q_+} d\xi' \rho_0(\xi') \sum_{l=1}^{2S}\frac{1}{\pi}\frac{l|c|}{(\xi-\xi')^2+(lc)^2}. \tag{37}$$

Except for the integration limits this equation is identical to Eq. (6). The integral equation for $\rho_1(\xi)$ is essentially Eq. (13) for $\Delta\rho_1(\xi,\xi_0)$, except for the integration limits. Finally, $\rho_2(\xi)$ is determined by the last two driving terms in Eq. (36).

Using the Euler–MacLaurin sum formula the energy of the system is given by [2,13]

$$E = L\int_{Q_-}^{Q_+} d\xi \epsilon^{(0)}(\xi)\rho_0(\xi) - \epsilon^{(0)}\left(\xi_L^h\right) + \int_{Q_-}^{Q_+} d\xi \epsilon^{(0)}(\xi)\rho_1(\xi)$$
$$- \frac{1}{24L}\left[\frac{\epsilon^{(0)'}(Q_+)}{2\pi\rho(Q_+)} - \frac{\epsilon^{(0)'}(Q_-)}{2\pi\rho(Q_-)}\right] + \frac{1}{L}\int_{Q_-}^{Q_+} d\xi \epsilon^{(0)}(\xi)\rho_2(\xi), \tag{38}$$

where $\epsilon^{(0)}(\xi) = (2S+1)[\xi^2 - S(S+1)c^2/3]$ and the prime denotes the derivative. The first term in Eq. (38) is $E_{GS} = L\epsilon_{GS}(Q_\pm)$, where $\epsilon_{GS}(Q_\pm)$ is the ground state energy density for the integration limits $Q_\pm$. The L^{-1}-terms reduce to $-\pi v_F/6L$ where v_F is the Fermi energy. Hence, the expression for the energy reduces to

$$E = L\epsilon_{GS}(Q_\pm) - \epsilon^{(0)}\left(\xi_L^h\right) + \int_{Q_-}^{Q_+} d\xi \epsilon^{(0)}(\xi)\rho_1(\xi) - \frac{\pi}{6L}v_F. \tag{39}$$

Following the procedure in Section 2.5, the second and third terms (of order L^0) reduce to the dressed energy of the hole but with integration limits $Q_\pm$ instead of $\pm Q$, $\varepsilon_h(\xi_L^h)$. The rapidity ξ_L^h also depends on the finite size of the system and can be expanded as $\xi_L^h = \xi^h + \delta\xi^h/L$, so that the impurity term is $-\varepsilon_h(\xi^h) - \varepsilon_h'(\xi^h)\delta\xi^h/L$. Again, the prime denotes the derivative with respect to ξ. The quantity $\delta\xi^h/L$ can be calculated, but it plays no relevant role in what follows.

Next we expand $\epsilon_{GS}(Q_\pm)$ to second order in $Q_\pm$ about $\pm Q$

$$\epsilon_{GS}(Q_\pm) = \epsilon_{GS}(\pm Q) + \sum_{\sigma=\pm} \left.\frac{\delta\epsilon_{GS}(Q_\pm)}{\delta Q_\sigma}\right|_{Q_\sigma=\sigma Q} (Q_\sigma - \sigma Q) + \frac{1}{2}\sum_{\sigma\tau} \left.\frac{\delta^2\epsilon_{GS}(Q_\pm)}{\delta Q_\sigma \delta Q_\tau}\right|_{Q_\sigma=\sigma Q; Q_\tau=\tau Q} \big[(Q_\sigma - \sigma Q)(Q_\tau - \tau Q)\big], \tag{40}$$

where the first term is the equilibrium ground state energy density and the second term vanishes, because $\delta\epsilon_{GS}/\delta\rho = 0$ defines the energy potential $\varepsilon(\xi)$ (Eq. (2)). Hence, the linear terms of the expansion vanish and the first corrections are quadratic. After lengthy algebra we obtain

$$\left.\frac{\delta^2\epsilon_{GS}(Q_\pm)}{\delta Q_\sigma \delta Q_\tau}\right|_{Q_\sigma=\sigma Q; Q_\tau=\tau Q} = \delta_{\sigma\tau} 2\pi v_F \big[\rho_0(Q)\big]^2. \tag{41}$$

The $\delta_{\sigma\tau}$ is a consequence of the independence of the two Fermi points [2,13].

The finite size corrections to the energy can be summarized as

$$E = L\epsilon_{GS}(\pm Q) - \varepsilon\big(\xi^h\big) + L\pi v_F \big[\rho_0(Q)\big]^2\big[(Q_+ - Q)^2 + (Q_- + Q)^2\big] - \pi v_F/6L - \varepsilon'\big(\xi^h\big)\delta\xi^h/L. \tag{42}$$

Next we have to relate the shifts of the integration limits with the quantum numbers of the high energy excitation of the system. To order $1/L$ the bound state density and current density are defined as

$$\frac{M}{L} = \frac{I_+ - I_-}{L} = \frac{1}{2\pi}\big[\mathcal{Z}(Q_+) - \mathcal{Z}(Q_-)\big] = \int_{Q_-}^{Q_+} d\xi\rho(\xi) = \int_{Q_-}^{Q_+} d\xi\rho_0(\xi) + \frac{1}{L}\int_{Q_-}^{Q_+} d\xi\rho_1(\xi),$$

$$\frac{2D}{L} = \frac{I_+ + I_-}{L} = \frac{1}{2\pi}\big[\mathcal{Z}(Q_+) + \mathcal{Z}(Q_-)\big] = \int_{-\infty}^{Q_+} d\xi\rho(\xi) - \int_{Q_-}^{\infty} d\xi\rho(\xi)$$

$$= \int_{-\infty}^{Q_+} d\xi\left[\rho_0(\xi) + \frac{1}{L}\rho_1(\xi)\right] - \int_{Q_-}^{\infty} d\xi\left[\rho_0(\xi) + \frac{1}{L}\rho_1(\xi)\right]. \tag{43}$$

We denote with N_{imp} and D_{imp} the quantities related to the high energy excitation (mobile impurity)

$$N_{imp} = \int_{Q_-}^{Q_+} d\xi\rho_1(\xi),$$

$$2D_{imp} = \int_{-\infty}^{Q_+} d\xi\rho_1(\xi) - \int_{Q_-}^{\infty} d\xi\rho_1(\xi). \tag{44}$$

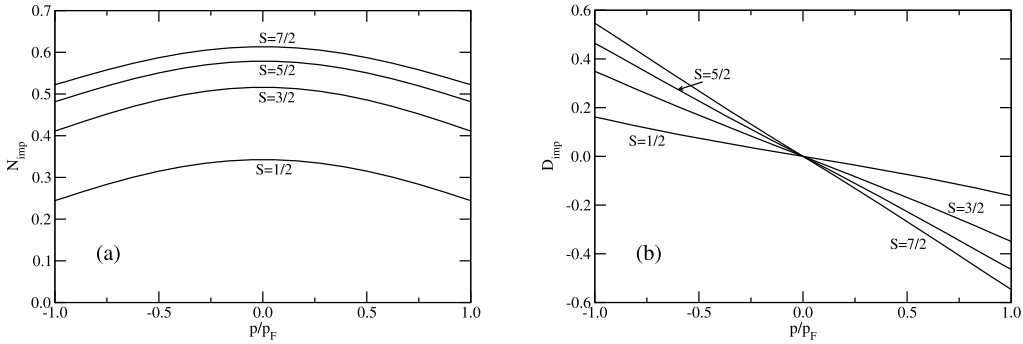

Fig. 3. (a) Bound state density N_{imp} and (b) current density D_{imp} for a high energy hole excitation as a function of the momentum of the excitation from Eq. (44) for $|c| = 1$, $Q = 1$ and several spin values. Note that N_{imp} is an even function and D_{imp} an odd function of the momentum.

The variation of the integration limits $Q_\pm$ with M/L to leading order in L^{-1} is given by

$$\frac{\partial Q_\pm}{\partial M/L} = \pm \frac{1}{2z\rho_0(Q_\pm)}, \tag{45}$$

where z is dressed generalized charge defined in Eq. (21). Similarly, the variation of the integration limits $Q_\pm$ with D/L is

$$\frac{\partial Q_\pm}{\partial D/L} = \frac{z}{\rho_0(Q_\pm)}. \tag{46}$$

It follows that

$$Q_\pm \mp Q = \pm \frac{1}{2z\rho_0(Q_\pm)} \left[\frac{\Delta M}{L} - \frac{N_{imp}}{L} \right] + \frac{z}{\rho_0(Q_\pm)} \left[\frac{\Delta D}{L} - \frac{D_{imp}}{L} \right] \tag{47}$$

which is inserted into Eq. (39) yielding

$$E = L\epsilon_{GS}(Q) - \varepsilon\left(\xi^h\right) - \frac{\pi}{6L} v_F - \varepsilon'\left(\xi^h\right)\delta\xi^h \frac{1}{L} + \frac{2\pi v_F}{L} \left[z^2 \tilde{D}^2 + \frac{(\Delta\tilde{M})^2}{4z^2} \right], \tag{48}$$

where $\Delta\tilde{M} = \Delta M - N_{imp}$ and $\tilde{D} = D - D_{imp}$ are the changes in the quantum numbers with respect to the ground state. N_{imp} and D_{imp} as a function of the momentum of the excitation are shown in Fig. 3 for a hole excitation and Fig. 4 for a particle excitation for several spin values. Note that N_{imp} is also given by $\partial\varepsilon(\xi^h)/\partial\mu$. Eq. (48) corresponds to a conformal field theory with central charge $c = 1$ and shifted quantum numbers [9,13].

We are now in condition to establish the relation between the field-theoretical approach and the Bethe *Ansatz*. From Eq. (30) we see that for the Luttinger liquid $K = z^2$, and for the mobile impurity the energy $\epsilon_h(p)$ is to be identified with $\varepsilon(\xi^h)$, while the impurity velocity u is given by Eq. (11) within the Bethe *Ansatz* framework.

Next we have to identify the scattering phase shifts within the two approaches, i.e. N_{imp} and D_{imp} with φ_+ and φ_-. Using Eq. (31) we have

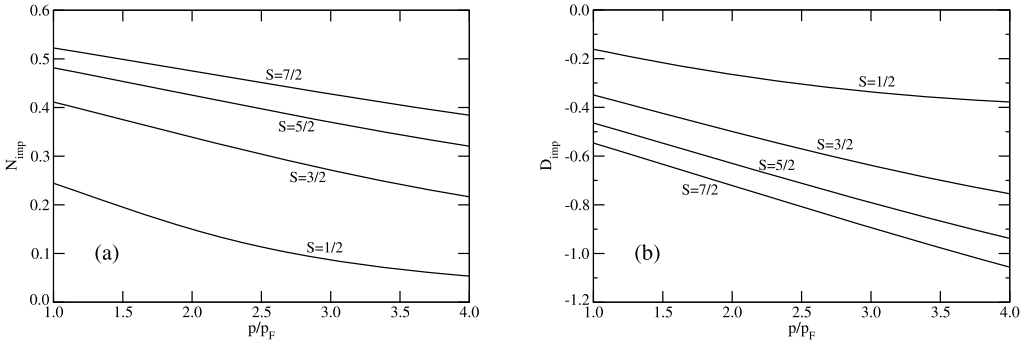

Fig. 4. (a) Bound state density N_{imp} and (b) current density D_{imp} for a high energy particle excitation as a function of the momentum of the excitation from Eq. (44) for $|c| = 1$, $Q = 1$ and several spin values. Note that N_{imp} is an even function and D_{imp} an odd function of the momentum.

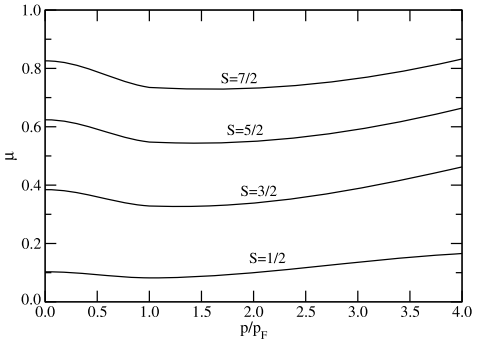

Fig. 5. Exponent μ of the one-particle (hole) excitation Green's function as a function of momentum for $Q = |c|$ and several spin values. The case $p < p_F$ corresponds to holes and $p > p_F$ to particles.

$$N_{imp} = \frac{1}{2\pi}(\varphi_- - \varphi_+),$$
$$2D_{imp} = -\frac{1}{2\pi K}(\varphi_+ + \varphi_-), \tag{49}$$

or

$$\varphi_\pm = \pi(\mp N_{imp} + K 2D_{imp}), \tag{50}$$

and the exponents are now given by Eq. (33). The correlation function for the hole (34) has now the correct energy threshold and the exact exponent for the singularity [9,11].

A similar calculation for an added particle yields a spectral function as in Eq. (32) with the correct energy threshold and exponent, Eq. (33). The exponent as a function of momentum of the excitation is shown in Fig. 5 for $Q = |c|$ and several spin values. The case $p < p_F$ corresponds to hole excitations and $p > p_F$ to particle excitations. $\mu(p)$ is an even function of p and a continuous function through the Fermi level at p_F.

Following the lines of Ref. [9], for an excitation close to the right Fermi point ($\xi^{p,h} \to Q$) one obtains

$$\frac{\varphi_+}{\pi} \to \mp(K-1), \qquad \frac{\varphi_-}{\pi} \to \pm\left(2\sqrt{K} - 1 - K\right), \tag{51}$$

where the sign refers to particles and holes. The exponent μ is then

$$\mu = \frac{1}{2K}\left(\sqrt{K}-1\right)^2(K+1), \tag{52}$$

which is believed to be an universal result.

Adding a low-energy particle to the right Fermi surface branch implies $u = v_F$ and according to conformal field theory this corresponds to $\Delta M = 1$ and $D = -1/2$, so that the one-particle Green's function is

$$G_p \sim e^{-i\epsilon t}\left[\frac{-i}{t-i\eta}\right]^{\nu}, \tag{53}$$

where

$$\nu = \frac{1}{4}\left(z-\frac{1}{z}\right)^2 = \frac{1}{K}\left(\frac{\varphi_+}{2\pi}\right)^2. \tag{54}$$

5. Conclusions

We considered a one-dimensional gas of fermions of spin S with an attractive δ-function interaction with possible application to ultracold atoms confined to an elongated trap. The particles form bound states of up to $N = 2S+1$ atoms, extending the concept of preformed Cooper pairs to larger clusters. These preformed bound states constitute a Luttinger liquid. We have studied the spectral function of high energy particle and hole excitations using a combination of field theory methods and the Bethe *Ansatz* solution. The time-dependence of the single particle Green's function can be obtained by an effective model consisting of the Luttinger liquid coupled to a mobile impurity, in analogy to other models investigated previously [5–13]. This constitutes an X-ray-edge-type model displaying the correct energy threshold and a power-law singularity with the correct exponent. As in Refs. [9,13] we justify the mobile impurity using the Bethe *Ansatz* solution of the model. This way the phenomenological parameters of the field theory model are determined from the Bethe *Ansatz*. These parameters are the Luttinger liquid parameter K, which is related to the generalized dressed charge z, the exact energy of the excitation, and the momentum dependent scattering phase shifts.

The procedure consists in using the discrete Bethe *Ansatz* equations to evaluate the $\mathcal{O}(1/L)$ corrections to the energy, i.e. the finite size corrections, for the system in the ground state and in a state with one high-energy particle or hole excitation. This extends the standard finite size terms for low-energy excitations about the Fermi points in a Luttinger liquid (conformal towers) to arbitrary excitations. Hence, the present approach goes beyond conformal field theory and the bosonization of fermions [1]. The high energy excited state is parametrized as a mobile impurity with the exact excitation energy and the linear coupling to the Luttinger liquid is analogous to the X-ray-threshold problem [14,15] and the power-law singularity in frequency or time is then a consequence of Anderson's "infrared orthogonality catastrophy" [63].

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